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Gaussian Bounds for Correlations in Lattice Spin Systems

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In quite general N-component ferromagnetic spin systems, it is proved that an arbitrary correlation function is bounded by the corresponding correlation function of a Gaussian model. The bound is useful for the analysis of high-temperature behavior of the system. Similar bounds for truncated correlation functions are also obtained for a class of single-component spin systems.

KEY WORDS: Rigorous inequalities; Gaussian model; clustering properties; truncated correlation functions.

1. INTRODUCTION AND MAIN RESULTS

The existence of a trivially solvable Gaussian model[1] has produced unspeakable benefits to the study of spin systems. The Gaussian model provides us with a qualitatively satisfactory picture of critical phenomena, and in various systems, one can estimate the deviation from the Gaussian behavior by suitable perturbative methods. Moreover, many rigorous results for non-Gaussian systems have been obtained by comparing the relevant systems with the Gaussian model.[2–5]

Here, for a very general class of ferromagnetic multicomponent spin systems, I prove that an arbitrary correlation function is bounded by that of a Gaussian model with the same Hamiltonian. And for a class of single-component spin systems, I prove similar bounds for truncated correlation functions.

Let Λ be a finite *lattice* (a set of *sites*) of arbitrary geometry. To each site $x \in \Lambda$, we equip N-component *spin variable* $\boldsymbol{\varphi}_x = \{\varphi_x^{(i)}\}_{i=1,\ldots,N}$, $\varphi_x^{(i)} \in \mathbb{R}$.

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The *thermal expectation* and the *partition function* of the system are defined as

$$\langle \cdots \rangle = Z^{-1} \int \prod_{x \in \Lambda} g(|\boldsymbol{\varphi}_x|^2) \prod_{i=1}^{N} d\varphi_x^{(i)} (\cdots) \, e^{-\mathscr{H}} \tag{1.1}$$

$$Z = \int \prod_{x \in \Lambda} g(|\boldsymbol{\varphi}_x|^2) \prod_{i=1}^{N} d\varphi_x^{(i)} \, e^{-\mathscr{H}} \tag{1.2}$$

where

$$|\boldsymbol{\varphi}_x|^2 = \sum_{i=1}^{N} (\varphi_x^{(i)})^2$$

The Hamiltonian is

$$\mathscr{H} = -\sum_{x,y,i} J_{xy}^{(i)} \varphi_x^{(i)} \varphi_y^{(i)} - \sum_{x,i} H_x^{(i)} \varphi_x^{(i)} \tag{1.3}$$

where *interactions* $\{J_{xy}^{(i)}\}$ and *external fields* $\{H_x^{(i)}\}$ are arbitrary nonnegative constants. *Single-site distribution function* $g(s)$ is a nonnegative valued function defined on $s \geqq 0$, which is normalized

$$\int \prod_{i=1}^{N} d\varphi^{(i)} \, g(|\boldsymbol{\varphi}|^2) = 1$$

and satisfies $\exp(ks)\, g(s) \to 0$ as $s \to \infty$ for any real constant k. In particular, we consider the following two classes of $g(s)$.

(i) $g(s)$ is an arbitrary (generalized) function with bounded support, i.e., there exists a positive constant b, and $g(s) = 0$ for $s > b$. For example, $g(s) = \text{const} \cdot \delta(s-1)$ defines the classical $O(N)$ Heisenberg model.

(ii) $g(s) = \text{const} \cdot \exp[-V(s)]$, where $V(s)$ is a smooth function satisfying $V''(s) \geqq 0$ for all $s > a$, where a is a finite constant.[2] An example is the N-component φ^4 model defined by $V(s) = \alpha s + \lambda s^2$ (α real, $\lambda > 0$). We can also consider any well-defined limit of such $g(s)$.

For an arbitrary index set $A = \{a_x^{(i)}\}_{x \in \Lambda, i=1,\ldots,N}$ ($a_x^{(i)} = 0, 1, 2,\ldots$); we write

$$\varphi^A = \prod_{\substack{x \in \Lambda \\ i=1,\ldots,N}} (\varphi_x^{(i)})^{a_x^{(i)}} \tag{1.4}$$

Then we have the following theorem.

[2] This can be satisfied if $V(s)$ is a polynomial of s.

Theorem 1. To any single site distribution function $g(s)$ in class (i) or (ii), and to any positive integer N, there corresponds a positive finite constant[3] c. Consider the thermal expectation $\langle \cdots \rangle$ defined by Eqs. (1.1)–(1.3) with $g(s)$, and the Gaussian thermal expectation $\langle \cdots \rangle^{G,c}$ defined by the same equations with $g(s)$ replaced by const $\cdot \exp(-s/2c)$. Then for arbitrary index set A, we have

$$0 \leqq \langle \varphi^A \rangle \leqq \langle \varphi^A \rangle^{G,c} \tag{1.5}$$

provided that the Gaussian expectation $\langle \cdots \rangle^{G,c}$ is well defined.

It should be noted that if the spin system (1.1)–(1.3) has a sensible infinite volume limit,[(6,7)] standard convergence argument assures that Theorem 1 is still valid in the limit.

Theorem 1 (for vanishing external field) was derived by Sokal[(4)] for N-component spin systems ($N=1, 2, 3, 4$) where the Gaussian inequalities are known.[4] Since his proof relies on the correlation inequalities, it seems hard to extend the proof to other spin systems, nor to cases with nonzero external fields.[5] On the other hand, the present proof applies to quite general spin systems (arbitrary N, nonzero external field, and almost arbitrary single-site distributions) since it does not rely on the full correlation inequalities. Instead it makes use of a version of high-temperature expansion and "single-site Gaussian inequality." "Single-site Gaussian inequality' is a kind of correlation inequality stated in a non-interacting spin system, and its proof is considerably easier than that of the usual correlation inequalities. Therefore, the present proof is formally applicable to a spin system with arbitrary interactions, such as Z_2-lattice gauge theories. But in this case, the corresponding Gaussian model (i.e., the $N=1$ Weingarten model[(10)]) is, unfortunately, always ill defined.[(10,11)] (See Remark 3 in Section 3.)

To see how the above theorem works,[(4)] consider an N-component spin system on a d-dimensional hypercubic lattice (i.e., $\Lambda = Z^d$) with any single site distribution function $g(s)$ in the classes (i) or (ii). A translation invariant Hamiltonian is defined by Eq. (1.3) with $J_{xy}^{(i)} = J\delta_{|x-y|,1}$, $J \geqq 0$, and $H_x^{(i)} = H^{(i)} \geqq 0$.

According to Section 3 [Eqs. (3.6)–(3.8)], we choose a constant c, and define a Gaussian model with a single-site distribution function $\exp(-s/2c)$, and the same Hamiltonian as above. It is easily observed that this Gaussian model is well defined for $J < (2dc)^{-1}$ and arbitrary $H^{(i)} \geqq 0$.

[3] Determination of the constant c is discussed in the proof. See Section 3.

[4] Specific (but stronger) results can be seen in Refs. 8 and 9.

[5] Sokal also proves Eq. (1.5) for nonzero H, if $|A| = \sum_x a_x = 1$ and the system is in the GHS-class ($N=1$).

Thus is this region, we can apply Theorem 1 to bound all the correlations by the corresponding Gaussian correlations. For magnetization, we have

$$m^{(i)} = \langle \varphi_x^{(i)} \rangle \leqq \frac{H^{(i)}}{(2c)^{-1} - dJ}$$

which recovers the mean-field bound for magnetization[12–16,25] in the first order of $H^{(i)}$. This immediately implies that the *spontaneous magnetization is vanishing* in this region. Next, if we set $H^{(i)} = 0$, all the n-point functions exhibit the *exponential clustering properties*

$$\langle \varphi_{x_1}^{(i_1)} \varphi_{x_2}^{(i_2)} \cdots \varphi_{x_n}^{(i_n)} \rangle \to 0 \qquad \text{as} \quad |x_i - x_j| \to \infty$$

These two properties imply the following simple lower bound for the inverse critical temperature

$$J_c \geqq (2dc)^{-1}$$

For the $O(N)$-Heisenberg model $[g(s) = \text{const} \cdot \delta(s-1)]$, we have [from Eq. (3.7)] $c = 1/N$ and the above bound is nothing but the well-known mean-field bound[17–20]

$$J_c \geqq N/2d$$

Proof of Theorem 1 is given in the following two sections. In Section 2, we review a version of high-temperature expansion[8,20,21] which is quite useful for our proof. Then in Section 3 we introduce single site Gaussian inequality and prove the theorem. In Section 4, we prove bounds similar to Theorem 1 for truncated correlation functions $\langle \varphi^A; \varphi^B \rangle = \langle \varphi^A \varphi^B \rangle - \langle \varphi^A \rangle \langle \varphi^B \rangle$ in a class of single-component systems.

2. RANDOM LOOP–RANDOM WALK REPRESENTATION

Here we use an expansion method and interpret our spin system into a system of (interacting) random loops and random walks [Eqs. (2.11), (2.12)]. For simplicity, we discuss in detail the simplest case with single component spin variables (i.e., $N = 1$), and vanishing external field (i.e., $H_x = 0$).

Let us expand the exponential functions in Eqs. (1.1), (1.2) into the following Taylor series:

$$e^{J_{xy}\varphi_x\varphi_y} = \sum_{n=0}^{\infty} \frac{(J_{xy})^n}{n!} (\varphi_x \varphi_y)^n$$

Substituting this expansion into Eq. (1.2), we obtain an expression for the partition function

$$Z=\int \prod_{x\in\Lambda} g(\varphi_x)\, d\varphi_x \prod_{x,y} e^{J_{xy}\varphi_x\varphi_y}$$

$$= \sum_{\{n_{xy}\}} \left\{ \prod_{x,y} \frac{(J_{xy})^{n_{xy}}}{n_{x,y}!} \prod_x \langle \varphi_x^{\sum_z n_{xz}} \rangle_0 \right\} \tag{2.1}$$

where in the first summation, each n_{xy} independently runs over non-negative integers. We introduced single-site expectation

$$\langle \cdots \rangle_0 = \int g(\varphi)\, d\varphi(\cdots) \tag{2.2}$$

and used a convention $n_{xy}=n_{yx}$.

Since the single-site expectation (2.2) is invariant under the change of variable $\varphi \to -\varphi$, the last factor in Eq. (2.1) is nonvanishing only if $\sum_z n_{xz}$ is even for all x. Following Aizenman,[21] we regard $\mathfrak{m}=\{n_{xy}\}$ as *flux numbers* of a random current on the lattice (see Fig. 1), and define its *source set* by

$$\partial\mathfrak{m}=\left\{ x\in\Lambda \;\middle|\; \sum_z n_{xz} \text{ is odd} \right\}$$

Then Eq. (2.1) can be written as

$$Z=\sum_{\partial\mathfrak{m}=\varnothing} \left\{ \prod_{x,y} \frac{(J_{xy})^{n_{xy}}}{n_{xy}!} \prod_x \langle \varphi_x^{\sum_z n_{xz}} \rangle_0 \right\} \tag{2.3}$$

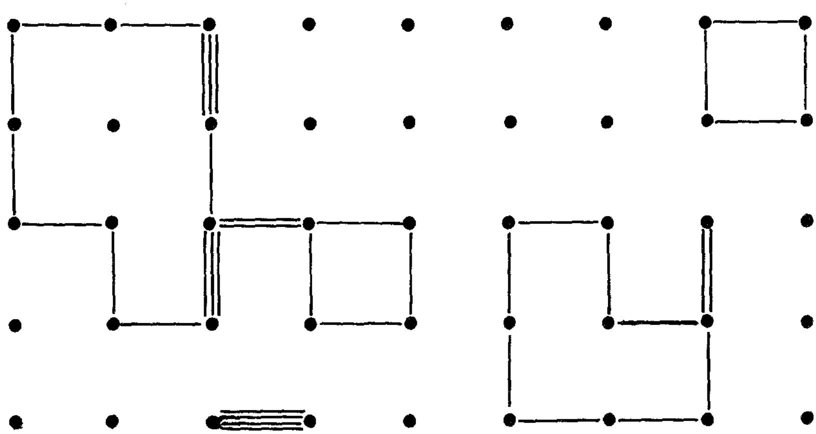

Fig. 1. A random current configuration which contributes to the partition function. Number of segments on a bond corresponds to flux number n_{xy}.

Similarly we can express an arbitrary correlation function by random currents:

$$\langle \varphi^A \rangle Z = \sum_{\partial \mathrm{m} = \partial A} \left\{ \prod_{x,y} \frac{(J_{xy})^{n_{xy}}}{n_{xy}!} \prod_x \langle \varphi_x^{\sum_z n_{xz} + a_x} \rangle_0 \right\} \tag{2.4}$$

where ∂A is a set of sites with odd a_x.

For a while, we restrict ourselves to the Gaussian model characterized by $g(s) = \text{const} \cdot \exp(-s/2c)$. Then the single-site expectation values in Eqs. (2.3), (2.4) can be explicitly evaluated as

$$\langle \varphi^{2m} \rangle_0^{G,c} = (2m-1)!!\, c^m \tag{2.5}$$

Note that the quantity $(2m-1)!!$ represents the number of ways of dividing $2m$ objects into m pairs. Accordingly, in the random current representations (2.3), (2.4), we divide $\sum_z n_{xz}$ currents attaching to site x into $(\sum_z n_{xz}/2)$ pairs, and regard each two currents in a pair as connected to each other. If we redefine the summation over the current configurations to count all the ways of pairing *repeatedly*, each factor $(\sum_z n_{xz} - 1)!!$ on the site can be absorbed in more detailed countings as in Fig. 2. Moreover if we do not distinguish a current from another one on the same bond (i.e., if we look only at the topology of the resulting graph), each factor $1/n_{xy}!$ on the bond also cancels out (see Fig. 3).

Consequently, the expression for the partition function (2.3) of the Gaussian model can be rewritten in a simple form

$$Z = \sum_{\{l\}} \prod_l f(l) \tag{2.6}$$

where the first summation runs over all the possible configurations of (arbitrary numbers of) *random loops*. A random loop l with length $|l| = n$ is a set of lattice bonds of the form

$$l = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_0)\}$$

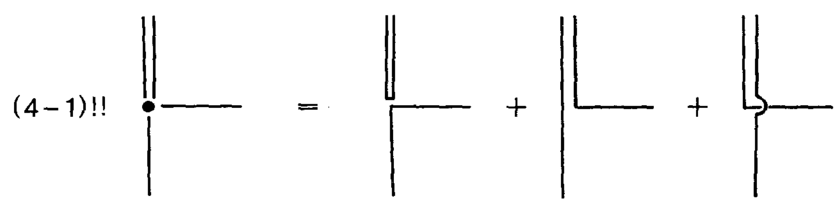

Fig. 2. Cancellation of a factor on a site.

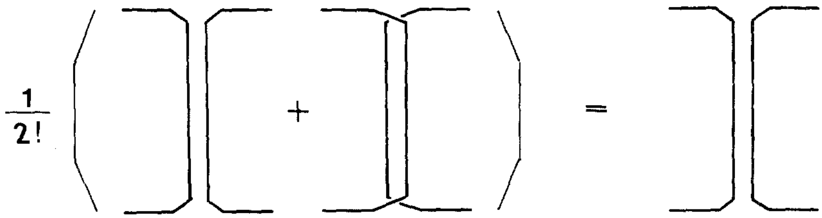

Fig. 3. Cancellation of a factor on a bond.

where x_i are arbitrary lattice sites (with possible duplications) satisfying $x_i \neq x_{i+1}$. Two such sets l, l' represent the same loop if and only if $x_i = x'_{i+k} (\mathrm{mod}\ k)$ holds for some k.

The non-negative weight factor $f(l)$ has the simple form

$$f(l) = J^l C^{|l|} \times \frac{1}{2^s}, \qquad s \in \{0, 1, 2, \ldots\}$$

where $J^l = J_{x_0 x_1} J_{x_1 x_2} \cdots J_{x_{n-1} x_0}$. The symmetry factor $1/2^s$ is not relevant to the present analysis. (The constant s is nonzero if and only if l is a loop consisting of duplicated copies of a curve.) Equation (2.6) can be interpreted as representing *a system of noninteracting random loops.*

Introducing a *random walk* ω with length $|\omega| = n$ connecting $x = x_0$ and $y = x_n$

$$\omega = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\}$$

a correlation function can also be represented by *noninteracting random loops and random walks* (Fig. 4)

$$Z \cdot \langle \varphi_{x_1} \varphi_{x_2} \cdots \varphi_{x_{2n}} \rangle^{G,c} = \sum_{\substack{\text{all pairings} \\ (x_1, \ldots, x_{2n}) \\ \downarrow \\ (\{p_1, q_1\}, \{p_2, q_2\}, \ldots, \{p_n, q_n\})}} \sum_{\substack{\omega_i: p_i \to q_i \\ (i = 1, \ldots, n)}} \sum_{\{l\}} \prod_i f(\omega_i) \prod_l f(l) \tag{2.7}$$

where $f(\omega) = J^{\omega} c^{|\omega| + 1}$ and the second summation runs over all possible random walks connecting p_i and q_i.

If we divide Eq. (2.7) by Eq. (2.6), all the contributions from random loops cancel and we recover the following well-known expression for the correlation function of the Gaussian model:

$$\begin{aligned} \langle \varphi_{x_1} \varphi_{x_2} \cdots \varphi_{x_{2n}} \rangle^{G,c} &= \sum_{\substack{\text{all pairings} \\ (x_1, \ldots, x_{2n}) \\ \downarrow \\ (\{p_1, q_1\}, \ldots, \{p_n, q_n\})}} \sum_{\substack{\omega_i: p_i \to q_i \\ (i = 1, \ldots, n)}} \sum_i f(\omega_i) \\ &= \sum_{\text{all pairings}} \langle \varphi_{p_1} \varphi_{q_1} \rangle^{G,c} \cdots \langle \varphi_{p_n} \varphi_{q_n} \rangle^{G,c} \end{aligned} \tag{2.8}$$

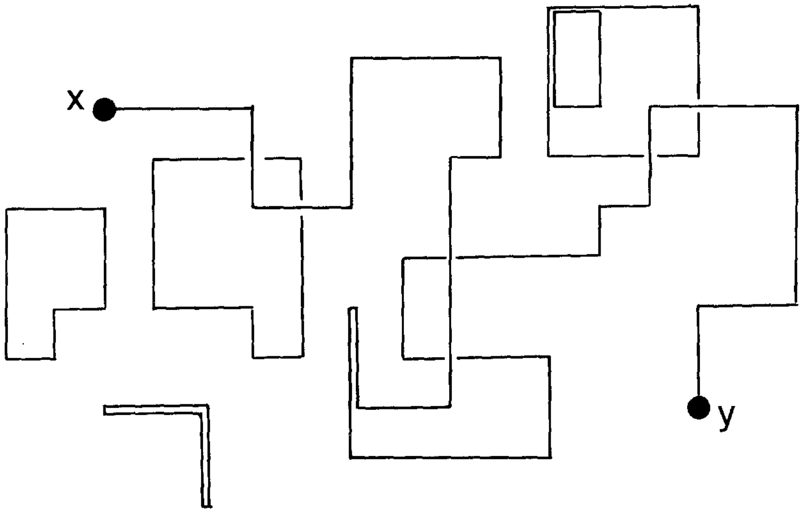

Fig. 4. A random loop–random walk configuration for correlation function $\langle \varphi_x \varphi_y \rangle$.

In general non-Gaussian systems, we have no closed expression for single-site expectations like Eq. (2.5). To execute a combinatorial transformation similar to that we have done in the Gaussian model, we formally rewrite Eq. (2.3) as

$$Z = \sum_{\partial \mathfrak{m} = \varnothing} \left\{ \prod_{x,y} \frac{(J_{xy})^{n_{xy}}}{n_{xy}!} \prod_x \left(\sum_z n_{xz} - 1\right)!!\, c^{(\sum_z n_{xz}/2)} \right\} \prod_x I\left(\frac{\sum_z n_{xz}}{2} \right) \tag{2.9}$$

where c is an arbitrary positive constant, and the extra factor $I(m)$ is defined by

$$I(m) = \frac{\langle \varphi^{2m} \rangle_0}{(2m-1)!!\, c^m} \tag{2.10}$$

Then it is easy to obtain the expressions corresponding to Eqs. (2.6) and (2.7).

$$Z = \sum_{\{l\}} \left\{ \prod_l f(l) \prod_x I(m_x) \right\} \tag{2.11}$$

$$Z\langle \varphi_{x_1} \cdots \varphi_{x_{2n}} \rangle = \sum_{\text{all pairings}} \sum_{\substack{\omega_i:\, p_i \to q_i \\ (i=1,\ldots,n)}} \sum_{\{l\}} \left\{ \prod_i f(\omega_i) \prod_l f(l) \prod_x I(m_x) \right\} \tag{2.12}$$

where m_x denotes the number of currents passing through the site x. It is very natural to interpret the terms with $I(m_x)$ as *the interactions between* *the random loops and random walks.* In the next section, we argue that for a suitable choice of the constant c, these interactions work to *avoid the intersections* between loops and walks. Because of these interaction terms, we have no formula corresponding[6] to Eq. (2.8).

To complete our discussions on the random loop–random walk representation, we have to extend the present formalism to multicomponent systems with nonvanishing external field.

To deal with the external field terms, we again expand the exponential:

$$e^{H_x \varphi_x} = \sum_{n=0}^{\infty} \frac{H_x{}^n}{n!} \varphi_x{}^n$$

Here n can be interpreted as a number of *external sources* which is graphically represented by "crosses" (Fig. 5). Then in the random loop–random walk representation, there appears a random walk terminated by crosses with extra factor H, and "single-site bubbles" consisting of $2m$ crosses and carrying a factor $(H^2c/2)^m/m!$ (Fig. 6). Correlation functions of the Gaussian model can again be written in terms of noninteracting random walks.

[6] Actually, it is possible to obtain a version of Eq. (2.8) by a further modification of the present scheme.[20,3]

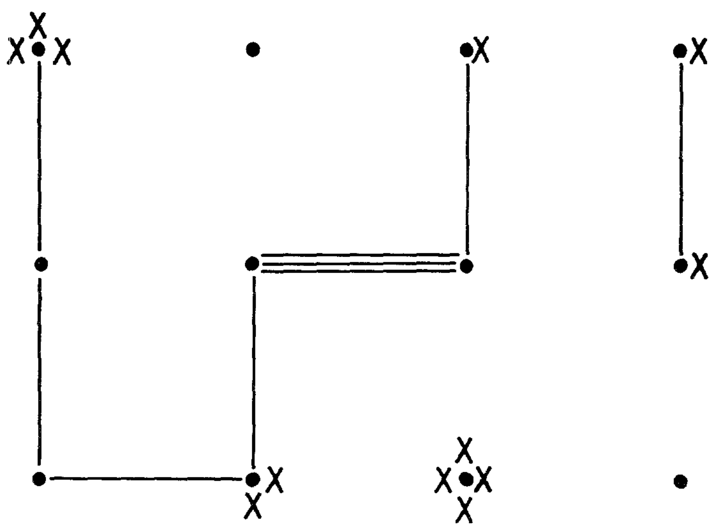

Fig. 5. A random current configuration with external field which contributes to the partition function.

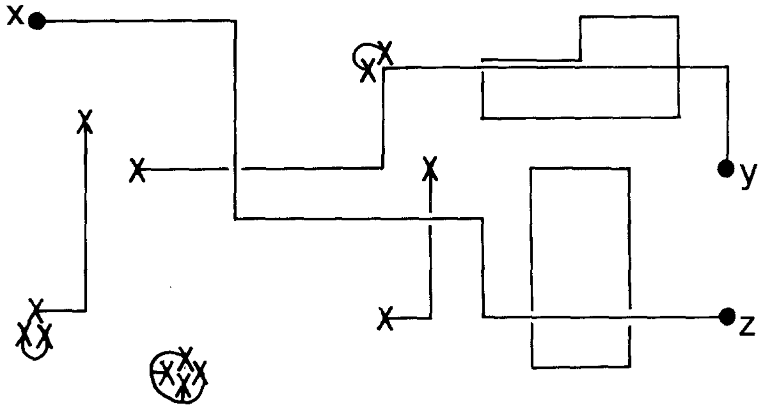

Fig. 6. A random loop–random walk configuration with external field for correlation function $\langle \varphi_x \varphi_y \varphi_z \rangle$.

For multicomponent spin systems, we note that the single-site expectation

$$\langle \cdots \rangle_0 = \int g(|\boldsymbol{\varphi}|^2) \prod_{i=1}^{N} d\varphi^{(i)} (\cdots) \tag{2.13}$$

is invariant under $\varphi^{(i)} \to -\varphi^{(i)}$ for any component i. Therefore, we can execute the expansions and combinatorial transformations independently in each component. Consequently we obtain random loops and random walks running in N different layers and interacting via the factor

$$I(\{m^i\}) = \frac{\langle \prod_{i=1}^{N} (\varphi^{(i)})^{2m^i} \rangle_0}{\prod_{i=1}^{N} (2m^i - 1)!! \, c^{\sum_{i=1}^{N} m^i}} \tag{2.14}$$

3. BOUNDS FOR CORRELATION FUNCTIONS

We now prove the Gaussian bound for arbitrary correlation functions (Theorem 1). The basic idea of the proof is to *cut off* the interactions between the random walks propagating the correlations and the other random loops and walks (Fig. 7).

Mathematically, this *cutting off* is done by the following *single-site Gaussian inequality.*

Lemma 2. To any single-site distribution function $g(s)$ in class (i) or (ii), and to any positive integer N, there corresponds a positive finite

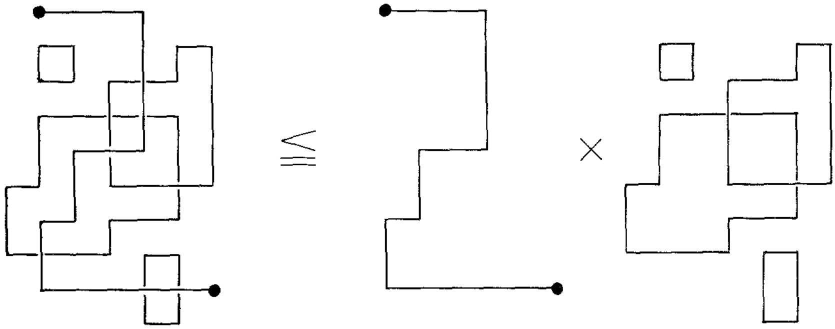

Fig. 7. The idea of bounding $\langle \varphi_x \varphi_y \rangle$ by $\langle \varphi_x \varphi_y \rangle^{G,c}$, when $N=1$ and $H_x=0$.

constant c. And for the corresponding single-site expectation[7] (2.13), we have

$$\left\langle \varphi_{(i)}^2 \prod_{j=1}^{N} \varphi_{(j)}^{2m_j} \right\rangle_0 \leqq (2m_i+1)\, c \left\langle \prod_{j=1}^{N} \varphi_{(j)}^{2m_j} \right\rangle_0 \tag{3.1}$$

for arbitrary $m_1, m_2, \dots, m_N \geqq 0$.

Proof of Theorem 1 given Lemma 2. We discuss the most general case with arbitrary N and nonzero external fields. First we set the arbitrary constant c which appears in the definition of $I(\{m^i\})$ [Eqs. (2.9), (2.10), and (2.14)] equal to that determined by Lemma 2. Consider the random loop–random walk representation of the quantity

$$Z\langle \varphi_{x_1}^{(i_1)} \varphi_{x_2}^{(i_2)} \cdots \varphi_n^{(in)} \rangle, \qquad x_j \in \Lambda, \qquad i_j = 1, \dots, N$$

In this representation, there appear two types of random walks. One consists of "background" random walks whose two ends are terminated by external field "crosses," and another type consists of the random walks terminated by (one or two) sites in the correlation function. We suppose that through a site x, $n_{\text{b.g.}}^{(i)}$ currents from random loops, "single-site bubbles," and "background" random walks are passing, and $n_{\text{corr}}^{(i)}$ currents from the second type random walks are passing, respectively, for each component i. Then applying the single-site Gaussian inequality (3.1) repeatedly at site x, we get

$$\left\langle \prod_{i=1}^{N} (\varphi_x^{(i)})^{2(n_{\text{b.g.}}^{(i)} + n_{\text{corr}}^{(i)})} \right\rangle_0$$
$$\leqq \left\{ \prod_{i=1}^{N} \frac{(2n_{\text{b.g.}}^{(i)} + 2n_{\text{corr}}^{(i)} - 1)!!}{(2n_{\text{b.g.}}^{(i)} - 1)!!} \right\} c^{\sum_{i=1}^{N} n_{\text{corr}}^{(i)}} \left\langle \prod_{i=1}^{N} (\varphi_x^{(i)})^{2n_{\text{b.g.}}^{(i)}} \right\rangle_0$$

[7] Here and in the proof of the lemma, we write $\varphi_{(i)}$ instead of $\varphi^{(i)}$ since we only discuss the single-site expectation.

which with Eq. (2.10) implies

$$I(\{n_{\text{b.g.}}^{(i)}+n_{\text{corr}}^{(i)}\}) \leqq I(\{n_{\text{b.g.}}^{(i)}\}) \tag{3.2}$$

Substituting Eq. (3.2) into random loop–random walk representation [corresponding to Eq. (2.12)], and using the representation for the partition function similar to Eq. (2.11) we obtain

$$\begin{aligned}
Z\langle \varphi_{x_1}^{(i_1)} \cdots \varphi_{x_n}^{(i_n)} \rangle
&= \sum_{\text{pairings}} \sum_{\{\omega\},\{l\},\{b\}} \left\{ \prod f(\omega) \prod f(l) \prod f(b) \prod_{x,i} I(n_{\text{corr},x}^{(i)} + n_{\text{b.g.},x}^{(i)}) \right\} \\
&\leqq \sum_{\text{pairings}} \left\{ \sum_{\{\omega\}} \prod f(\omega) \right\} \times \sum_{\{\omega'\},\{l\},\{b\}}' \left\{ \prod f(\omega') \prod f(l) \prod f(b) \prod_{x,i} I(n_{\text{b.g.},x}^{(i)}) \right\} \\
&= \sum_{\text{pairings}} \left\{ \sum_{\{\omega\}} \prod f(\omega) \right\} \times Z
\end{aligned}$$

From the formula (2.10) for correlation function of the Gaussian model, we get the desired Gaussian bound

$$0 \leqq \langle \varphi^A \rangle \leqq \langle \varphi^A \rangle^{G,c}$$

(The first inequality is the Griffiths inequality,[22,6] which is a straightforward consequence of the expansion scheme of the previous section.) ▮

Proof of Lemma 2. The proof makes use of the standard Fourier transformation and integration-by-parts method.[20,3] Regarding $g(|\boldsymbol{\varphi}|^2)$ as a function of $(\varphi_{(i)})^2$, $i=1,\ldots,N$, we introduce its Fourier representation

$$g(|\boldsymbol{\varphi}|^2) = \int \prod_{i=1}^{N} da_i \exp\left(-\sum_{i=1}^{N} ia_i \varphi_{(i)}^2 \right) \hat{g}(\{a_i\})$$

We consider Eq. (3.1) for $i=1$

$$\begin{aligned}
&\left\langle \varphi_{(1)}^2 \prod_{i=1}^{N} \varphi_{(i)}^{2m_i} \right\rangle_0 \\
&\quad = \int \prod_{i=1}^{N} d\varphi_{(i)}\, da_i\, \varphi_{(1)}^2 \prod_{i=1}^{N} \varphi_{(i)}^{2m_i} \exp\left(-\sum_{i=1}^{N} ia_i \varphi_{(i)}^2 \right) \hat{g}(\{a_i\})
\end{aligned}$$

Executing integrations-by-parts m_i times for each component, we have

$$= (2m_1+1)\prod_{i=1}^{N}(2m_i-1)!!\int\prod_{i=1}^{N}d\varphi_{(i)}\,da_i\cdot\varphi_{(1)}^2\prod_{i=1}^{N}\frac{1}{(2ia_i)^{m_i}}$$
$$\times\exp\left(-\sum_{i=1}^{N}ia_i\varphi_{(i)}^2\right)\hat{g}(\{a_i\})$$

Inserting the identity

$$\frac{1}{(2ia)^m}=\int_0^\infty\frac{e^{-2iat}t^{m-1}}{(m-1)!}dt \qquad (m\geqq 1) \tag{3.3}$$

and transforming back to non-Fourier representation, we get

$$= (2m_1+1)\prod_{i=1}^{N}(2m_i-1)!!\int\prod_{i=1}^{N}d\varphi_{(i)}\int_0^\infty\prod_{i=1}^{N}dt_i\frac{t_i^{m_i-1}}{(m_i-1)!}$$
$$\times\ \varphi_{(1)}^2\,g(\{\varphi_{(i)}^2+2t_i\})$$
$$= (2m_1+1)\prod_{i=1}^{N}(2m_i-1)!!\int\prod_{i=1}^{N}d\varphi_{(i)}\int_0^\infty\prod_{i=1}^{N}dt_i\frac{t_i^{m_i-1}}{(m_i-1)!}$$
$$\times\langle\varphi_{(1)}^2\rangle_{\Sigma t_i}\,g(\{\varphi_{(i)}^2+2t_i\}) \tag{3.4}$$

where we introduced single-site expectation modified by t:

$$\langle\cdots\rangle_t=\frac{\int\prod_{i=1}^{N}d\varphi_{(i)}\,g(|\boldsymbol{\varphi}|^2+2t)(\cdots)}{\int\prod_{i=1}^{N}d\varphi_{(i)}\,g(|\boldsymbol{\varphi}|^2+2t)}$$

In the next part of the proof, we show that constant c can be chosen to satisfy

$$\langle\varphi_{(1)}^2\rangle_t\leqq c \qquad \text{for any}\quad t\geqq 0 \tag{3.5}$$

Using this result, we obtain the desired Eq. (3.1)

$$\left\langle\varphi_{(1)}^2\prod_{i=1}^{N}\varphi_{(i)}^{2m_i}\right\rangle_0$$
$$\leqq (2m_1+1)\,c\prod_{i=1}^{N}(2m_i-1)!!\int\prod_{i=1}^{N}d\varphi_{(i)}\int_0^\infty\prod_{i=1}^{N}dt_i\frac{t_i^{m_i-1}}{(m_i-1)!}g(\{\varphi_{(i)}^2+2t_i\})$$
$$= (2m_1+1)\,c\left\langle\prod_{i=1}^{N}\varphi_{(i)}^{2m_i}\right\rangle_0$$

The last identity follows from the calculation similar to that used in deriving Eq. (3.4). ▮

Proof of Eq. (3.5) **How to determine the constant** c. We demonstrate that we can choose a finite constant c satisfying

$$c \geqq \max\{\langle \varphi_{(1)}^2 \rangle_t \mid t \geqq 0\} \tag{3.6}$$

We consider two classes of single-site distribution functions separately.

Class i (bounded spin systems). Since $g(s)$ is vanishing for $s > b$, we have

$$\langle \varphi_{(1)}^2 \rangle_t = \frac{1}{N} \langle |\boldsymbol{\varphi}|^2 \rangle_t = \frac{1}{N} \frac{\int d\boldsymbol{\varphi}\, g(|\boldsymbol{\varphi}|^2 + 2t)\, |\boldsymbol{\varphi}|^2}{\int d\boldsymbol{\varphi}\, g(|\boldsymbol{\varphi}|^2 + 2t)} \leqq \frac{1}{N}(b - 2t)$$

Thus $\max\{\langle \varphi_{(1)}^2 \rangle_t\}$ is bounded by b/N. We can set

$$c = b/N \tag{3.7}$$

Class ii $\{\exp[-V(s)]$ measure$\}$. We will prove that $d/dt \langle \varphi_{(1)}^2 \rangle_t \leqq 0$ for $t \geqq a/2$. (See Section 1 for the definition of the constant a.) Then it is clear that

$$\max\{\langle \varphi_{(1)}^2 \rangle_t \mid t \geqq 0\} = \max\{\langle \varphi_{(1)}^2 \rangle_t \mid a/2 \geqq t \geqq 0\} < \infty$$

In particular if $a = 0$ (i.e., $V''(s) \geqq 0$ for all $s \geqq 0$), we have

$$c = \langle \varphi_{(1)}^2 \rangle_0 \tag{3.8}$$

To prove the claimed inequality for the derivative, we note that

$$\frac{d}{dt} \langle \varphi_{(1)}^2 \rangle_t = \frac{1}{N} \frac{d}{dt} \langle |\boldsymbol{\varphi}|^2 \rangle_t = -\frac{2}{N} \frac{[\varphi^2 V'(\varphi^2 + 2t)][1] - [\varphi^2][V'(\varphi^2 + 2t)]}{[1]^2}$$

where

$$[\cdots] = \int_0^\infty \varphi^{N-1}\, d\varphi (\cdots)\, e^{-V(\varphi^2 + 2t)}$$

Using two independent variables φ, ψ and duplicated expectation

$$[[\cdots]] = \int_0^\infty \varphi^{N-1}\, d\varphi\, \psi^{N-1}\, d\psi (\cdots)\, e^{-V(\varphi^2 + 2t) - V(\psi^2 + 2t)}$$

we can rewrite the numerator as

$$\begin{aligned} &[[\varphi^2 V'(\varphi^2 + 2t) - \varphi^2 V'(\psi^2 + 2t)]] \\ &\quad = \tfrac{1}{2}[[(\varphi^2 - \psi^2)(V'(\varphi^2 + 2t) - V'(\psi^2 + 2t))]] \end{aligned}$$

Since $V'(s+2t)$ is nondecreasing in s for $t \geqq a/2$, the quantity $(1/2)[[\cdots]]$ is non-negative and we obtain the desired inequality. ∎

Remarks

(1) If $g(s)$ in class (i) can be obtained as a limit of some single-site distribution in class (ii) with $a=0$, one should use the formula (3.8) for the determination of c, which is always not worse than Eq. (3.7).

(2) As is clear from the proof, we can consider more general bounded spin systems where $g(\{\varphi_{(i)}\})$ is an arbitrary (generalized) function with compact support and invariant under $\varphi_{(i)} \to -\varphi_{(i)}$ for any i. This allows, for example, $O(N)$-clock models (discrete vector models) and the three-state Potts model.

(3) In the Ising models with $\langle \varphi^{2n} \rangle_0 = 1$, we can use the inequality $1/\{2(n+m)-1\}!! \leqq \{1/(2n-1)!!\}\{1/(2m-1)!!\}$ instead of Eq. (3.1) to obtain an improved upper bound in terms of random walks with $1/(2n-1)!!$ self interactions. This improvement yields well-defined upper bounds for the loop expectation values of Z_2-lattice gauge theories as well.[(26)]

4. BOUNDS FOR TRUNCATED CORRELATION FUNCTIONS

In the present section, we restrict ourselves to single-component spin systems (i.e., $N=1$). The single-site distribution functions in consideration are the whole of the class (i) and the following subset of class (ii):

(ii)′ $g(s) = \text{const} \cdot \exp[-V(s)]$, where $V(s)$ is a polynomial of s with nonnegative coefficients. Again any well-defined limit is also considered.

For any index sets A and B, we define a (once-) truncated correlation function by

$$\langle \varphi^A; \varphi^B \rangle = \langle \varphi^A \varphi^B \rangle - \langle \varphi^A \rangle \langle \varphi^B \rangle$$

Theorem 3. To any single-site distribution function $g(s)$ in the classes (i) or (ii)′, there corresponds a finite positive constant[8] $\tilde{c}$. For any index sets A and B, we have

$$0 \leqq \langle \varphi^A; \varphi^B \rangle \leqq \langle \varphi^A; \varphi^B \rangle^{G,\tilde{c}} \tag{4.1}$$

provided that the Gaussian expectation $\langle \cdots \rangle^{G,\tilde{c}}$ is well defined.

[8] The constant $\tilde{c}$ is determined in the proof [Eqs. (4.5) and (4.6)]. For spin-1/2 Ising model $g(s) = (1/2)\,\delta(s-1)$, we can set $\tilde{c} = c = 1$.

Theorem 3, for the case of $|A| = |B| = 1$, was also proved by Sokal[4] for any single-component system in the GHS-class (i.e., Ellis–Monroe–Newman class[24]). The present proof, unfortunately, does not cover whole of the GHS-class.

Again the above theorem is valid in any sensible infinite-volume limit. Using the theorem, one can prove stronger clustering properties

$$\langle \varphi^A; \varphi^{B+y} \rangle \to 0 \qquad \text{as} \quad |y| \to \infty$$

for arbitrary values of nonnegative external field if the interaction is sufficiently small (i.e., the temperature is sufficiently high). Here $B + y$ denotes $\{b_{x-y}\}_{x \in \Lambda}$ if B is $\{b_x\}_{x \in \Lambda}$.

Proof. The proof is done by combining the method developed in the preceding sections and the standard technique of duplicated variables.

Let $\{\varphi_x\}$ and $\{\psi_x\}$ be two independent sets of spin variables. Duplicated thermal expectation is defined by

$$\langle \cdots \rangle_{\text{dup}} = (Z_{\text{dup}})^{-1} \int \prod_x d\varphi_x \, d\psi_x \, g(\varphi_x^2) \, g(\psi_x^2)(\cdots) \, e^{-\mathscr{H}(\{\varphi\}) - \mathscr{H}(\{\psi\})} \quad (4.2)$$

Define as usual[23,7]

$$T_x = \frac{1}{\sqrt{2}} (\varphi_x + \psi_x), \qquad Q_x = \frac{1}{\sqrt{2}} (\varphi_x - \psi_x), \qquad x \in \Lambda \quad (4.3)$$

We regard $\{T_x\}$ and $\{Q_x\}$ as the first and second components, respectively, of our duplicated two-component spin system, and are going to prove the following Gaussian bounds for this system:

$$0 \leqq \langle T^A Q^B \rangle_{\text{dup}} \leqq \langle T^A Q^B \rangle_{\text{dup}}^{G,\tilde{c}} \quad (4.4)$$

Then using the well-known relation

$$\langle \varphi^A; \varphi^B \rangle = \langle \varphi^A \varphi^B - \varphi^A \psi^B \rangle_{\text{dup}} = \sum_{C,D} \text{const}(C, D) \langle Q^C T^D \rangle_{\text{dup}}$$

where $\text{const}(C, D) \geqq 0$, and noting that the Gaussian model is invariant under the transformation (4.3), we obtain the desired inequality (4.1).

To prove the asserted bound (4.4), we repeat every step in Sections 2 and 3 for $\{Q_x\}$, $\{T_x\}$ variables. The only difference is that the single-site distribution function

$$\tilde{g}(Q^2, T^2) = g(\varphi^2) \, g(\psi^2)$$

is generally not a function of of Q^2+T^2. Hence the constant $\tilde{c}$ must be determined to satisfy

$$\tilde{c} \geqq \max\{\langle Q^2\rangle_{t,t'} \mid t, t' \geqq 0\}$$

where

$$\langle\cdots\rangle_{t,t'} = \frac{\int dQ\, dT\, \tilde{g}(Q^2+2t, T^2+2t')(\cdots)}{\int dQ\, dT\, \tilde{g}(Q^2+2t, T^2+2t')}$$

For $g(s)$ in *class* (i), noting that $|\varphi|$ and $|\psi|$ are always bounded by a constant $b^{1/2}$, we have

$$\max\{\langle Q^2\rangle_{t,t'}\} \leqq \left[\frac{1}{\sqrt{2}}(b^{1/2}+b^{1/2})\right]^2 = 2b$$

So one can set

$$\tilde{c} = 2b \tag{4.5}$$

which is, by a factor of 2, worse than the nontruncated case, Eq. (3.7).

For $g(s)$ in *class* $(ii)'$, we can again show that $\langle Q^2\rangle_{t,t'} \leqq \langle Q^2\rangle_{0,0}$ for any nonnegative t and t'. Thus one can set

$$\tilde{c} = c = \langle\varphi^2\rangle_0 \tag{4.6}$$ ∎

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